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PRESIDENTIAL ADDRESS: MATHEMATICAL COMBINATIONS WHICH CORRESPOND TO PHYSICAL IMPOSSIBILITIES

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In speaking to you as a representative from the Mathematical Section of the Academy, I have a feeling of hesitance which is quite usual among mathematicians when they address a mixed or non-mathematical group of scientific men and women. It is a well established tradition that scientists may complacently separate themselves from mathematics and it is equally true that mathematicians are timid about discussing their work with scientists outside their own circle. On the one hand the scientist tends to avoid abstraction, and on the other hand the mathematician does not seek reality. It was this feeling of separation that prompted the late Alfred Pringsheim, in his presidential address before the Bavarian Academy of Science, to say, "mathematics enjoys an outstanding unpopularity."

Notwithstanding this situation, the phenomena of nature have always been a source of mathematical inspiration; and mathematics has often served as the tool by which the scientist has progressed in an understanding of nature. As early as twenty-five centuries ago, the Greek philosophers recognized the close relationship which exists between nature and mathematics. Modern science has done much to establish this relationship. The universe is orderly and logical, so is mathematics; the universe has form and shape, mathematics has geometry; the universe has distinct recognizable entities, mathematics has numbers. The correspondence between nature and mathematics is strikingly evident. The patterns found in nature correspond to the principles and laws of mathematics, and conversely the logical structure of mathematics has corresponding models in nature. As G. A. Birkhoff has said, "the two are as closely related as the two sides of a coin."

Mathematics has developed two distinct lines. One in which the advances have followed physical models or guides and the other in which logical extensions of existing mathematics have been created in the human mind. These two separate points of view have tended to divide the subject into two rather indefinitely defined parts generally designated as applied mathematics and pure mathematics. Both lines have been fruitful. The mathematical treatment of physical phenomena has led to the invention of new operators along with the definition of new mathematical quantities which have been studied and extended so as to produce new and highly important mathematical theories, many of which seem at the time to have no rational connection with nature. However, in many instances these combinations which at first seem to suggest impossible physical situations have been of use to the physicist in pointing out new fields of research.

To a somewhat lesser degree than in the physical sciences, mathematics bears a similar relation to the biological and social sciences. Many of the laws in these highly observational sciences have been mathematically formulated and starting from these laws new and interesting mathematical results have been obtained. Our modern statistical theories have been built almost exclusively according to patterns existing in the biological and social sciences. Names such as Mendel, Galton, and Pearson are equally familiar to the biologist and the mathematician. Likewise Bernoulli, Laplace, Cournot, and Irving Fisher are known to both the mathematician and the social scientist.

Nor has mathematics been wholly separated from the fine arts, music, painting, sculpture, and poetry. Human emotions and esthetic principles apparently have mathematical properties which are much more than accidental although as yet in most cases the relationship is not fully defined.

We may assume that the role of mathematics is to give to nature a language and operations suitable for defining the various forms which may be observed, this is equivalent to saying that nature is mathematically ordered. The phenomena of the universe are in correspondence with logically derived mathematical theorems. Under this assumption it is evident that present day mathematics is at times redundant and at other times deficient. There are a great many elegant theorems in mathematics which have no known physical models and likewise there are an almost countless number of physical patterns which have not been reduced to a mathematical basis. However, no one can draw a clear line between the classes of mathematical theorems which have physical significance and those which do not.

The principle of universality has been the direct inspiration of much progress in mathematics. For example, if we assume that all quadratic equations have solutions, we must at once begin to consider numbers whose squares are negative numbers. In order that all quadratic equations may be solved, we set up the broad class of complex numbers; that is, numbers of the form $a + bi$ where a and b are real and i is the square root of minus one. The imaginary numbers, as the name implies, do not represent any physical entities; they exist only in the human mind. Notwithstanding they have been of great importance, in an operational way, in the interpretation of physical phenomena and as a tool in the hands of the research worker in applied mathematics. The theory of functions of a complex variable with its powerful methods of analysis has been exceedingly potent. The general procedure has been to take the results in the field of complex numbers, which is highly operational in character but which taken as a whole does not represent physical patterns, and extract from them the portions that may be interpreted in the field of application. Integrals, derivatives, limits and other devices in the domain of complex imaginary numbers may be interpreted in the rational domain of physics. The extension of the concept of real numbers into the broader field of the complex numbers, although physical impossibilities are involved, gives highly important interpretative results.

In pursuing mathematical paths suggested by nature, one finds many processes that depend essentially on positive integers. The exponents of powers, the order of derivatives, the number of times a function may be integrated are examples of this kind. Thus if we define the exponent of a power as a number which indicates the number of times a base is taken as a factor only positive integers have a meaning as exponents. However, the extension of exponents to a larger class of numbers, to the rational or perhaps to the real numbers, suggests itself at once. On the assumption that the laws of exponents which are known to hold for positive integers also hold for any number type, we are able to redefine the meaning of exponents in such a manner that positive and negative rational numbers, real numbers, and even complex numbers may have a perfectly well-defined meaning. The meaning of a with the exponent $-n$ and a with the exponent p divided by ν is perfectly clear as a mathematical quantity.

In a similar manner the order of a derivative has been defined so as to include fractions and negative numbers. One definition depends on Cauchy's integral in the complex plane and the other on what is generally known as the Leibnitz formula for the higher derivatives of a product of two functions. With either definition the use of fractions or negative numbers to give the order of derivatives offers no logical difficulties and the same definition serves for both derivatives and integrals, the positive orders are derivatives and the negative orders are integrals. I do not believe that fractional derivatives have as yet been used to handle physical problems; but the negative ones have considerable use especially in the solution of non-homogeneous linear differential equations. Velocities, slopes, curvature and other such physical concepts apparently limit the interpretation of derivatives to those of integral orders, and lengths, areas, volumes, masses, centroids, movements of intertia and similar quantities are expressed by integrals iterated an integral number of times. However, the physically impossible orders of derivatives and integrals offer no serious mathematical difficulties and the continuity of values which the order number may assume simplifies some of the mathematical analysis.

There are a great many mathematical operations of such a nature that when they are used with a given type of number the result of the operation is a more general kind of number than the original number. For example, if we apply the fundamental operations addition and multiplication and their inverses, subtraction and division, to positive integers, we may or may not obtain positive integers. The sum or the product of two positive integers is always a positive integer; but if we subtract a positive integer from another positive integer or divide a positive integer by another positive integer, we may easily obtain a result which lies outside the field of positive integers in the field of negative integers and fractions. In this manner we extend the notion of number from the field of positive integers and set up the field of all rational numbers, both positive and negative. In such an extended field the four fundamental operations of arithmetic may be performed and the result is always a number belonging to the field, with the single exception of division by zero. That is, the rational numbers are a closed number field with respect to the fundamental operations of arithmetic. Sums, product, differences, and quotients of rational numbers are rational numbers.

It is not at all difficult to find physical quantities which correspond to positive rational numbers, but at first glance we may think that the use of negative numbers leads to a physical impossibility. Negative masses, negative forces, and other negative physical entities do not seem to have any existence. But the logical extension of the arithmetical operations from positive to negative rational numbers can be readily interpreted. The mathematical symbols for negative numbers have a ready correspondence with many physical quantities. Forces may be designated as positive or negative according to their direction. A source makes use of the positive sign and a sink of a negative sign. We have positive and negative changes of electricity, profits are positive and the red figures showing losses correspond to negative numbers. In this manner a new framework has been set up to describe very practical situations.

When a physical model has been used as a framework for establishing new mathematical theories expressible by means of a symbolic formula or equation, as for example an integral or a differential equation, the evaluation of the integral or the solution of the equation will in most cases give results of a much more extensive character than is necessary to explain the original phenomena. Some of these new results may be interpreted as new results in the physical field of the original model, but some of the results of the mathematical process are clearly impossible in nature without a redefinition of the correspondence between the mathematical symbols and the physical entities. For example, let us assume that a certain type of equation has for its solution the number of times a cannon can be fired before reboring. Evidently such an equation would show a relation between the number and the caliber, muzzle velocity, charge, weight of projectile, and other ballistic constants belonging to the gun. If the equation actually represents this result the solution should be a positive integer, since no gun can be fired a negative or a fractional number of times. Other results of a similar character may appear although such solutions are impossible for the purpose of simple enumeration. They can be given significance as for example in the expressions of averages or departures from such averages.

The mathematical concept of infinity represents a physical impossibility, infinite forces, infinite masses, infinite distances and infinite time are not found in the laboratory. Infinity has its real existence in the human mind. By means of various devices the mathematician has been able to harness the concept and bring it into the logical structure of mathematics in a productive manner. Consider for example the concept of a Green's function, a function which on the boundary of a given area takes on given values and on the interior satisfies Laplace's differential equation except on one point at which it becomes infinite in a particular manner. Such a function does not correspond to any physical model and at the point where it becomes infinite is meaningless. In fact the Green's function is something like a fictional character inserted in the picture for the purpose of displaying certain physical results. However, when brought into a convenient operational relationship with other functions, it becomes possible to determine the value of the useful function at the point for which the Green's function becomes infinite. In other words the wild and physically impossible function makes it possible to determine the unknown but very respectable function.

The fact that the physical world is three-dimensional is to modern mathematics only incidental. If space were four-or-more-dimensional, the mathematical treatment would, to a marked degree, be at hand. Of course not all the details would be complete, but the fundamental features would at once be subject to objective mathematical treatment and many of the details could readily be worked out. The analysis of four-dimension space is essentially a study of functions of four variables and of dimensional space a study of functions of n variables. The great advantage of the four or more dimensional idea lies in the ability to translate the results known to exist in three-dimensional space over into the space of higher dimensions. For example if the variable t, for time, be associated with the three-dimensional space coordinates, x, y, and z, we have in fact a four-dimension space which has a correspondence of no little use in the field of relativity.

The remarkable advances in an understanding of the role of sampling in modern statistics has been to a large extent due to the mathematical treatment of a n-dimensional space. The logical development of such a space, which although physically non-existent, has served a guide for the development of statistical theories in sampling, in particular by R. A. Fisher and his school. The sizes or measures of the variates of a sample are used to locate a point in the super physical space and the conformations formed by points of this kind have served as models for the statistician in his studies and interpretation of the effects of sampling a population.

The extension of mathematical deductions into fields which are apparently separated from physical realities may be continued almost indefinitely. That which I have chosen to call a physical impossibility is not a barrier at the end of a path but rather a challenge to review the steps which have been taken and to examine the nature of the correspondence of the symbols and operations of mathematics to the physical phenomena. The following is a summary of the manner in which the situation may be met:

1. The interpretation of the mathematical results may depend on a broader definition of the correspondence involved and more detailed analysis of the physical entities may indicate properties which can be readily handled by the otherwise extraneous mathematics. For example in rectilinear motion, positive numbers are sufficient to express the speed. The use of negative numbers makes possible a fuller characterization of the motion by means of the velocity.

2. The mathematical symbols and operations which show an obvious lack of physical meaning may be used as a catalytic agent to

produce other results of importance. The use of imaginary complex numbers is an outstanding example of this method.

3. By considering a class of physical entities or phenomena as a whole mathematical quantities and processes, not applicable to the units themselves, have correspondence with the class as a whole. This is the underlying principle back of the statistical method in science.

4. The mathematical conventions without immediate physical applications may be used as a pattern for the more extensive investigations of physical situations. The n-dimensional fiction has been very useful in this manner.

5. The universality of pure mathematics and the logical completeness which is obtained through the establishment of theories of a wholly abstract character frequently enriches the field of applied mathematics and increases its power to discover and explain physical models. The realism of applied mathematics has a marked dependence on abstract reasoning.

Of course no one expects that all the logical deductions of mathematics shall become applicable in some scientific situation. The history of mathematics has shown that many years, even as many as a thousand, have passed before some mathematical truth has been rescued from oblivion and put to practical purpose, and moreover much of the existing mathematics will never be used in the explanation of nature.

The establishment of the correspondence between mathematics and nature involves a two-fold responsibility. There must be full and complete cooperation between the mathematical and non-mathematical scientists. The mathematician must undertake to give at least a part of his energies to making the rich symbolism and fruitful processes of mathematics available for his scientific colleagues. Research in pure mathematics cannot be discounted in any manner whatever, since the discovery of truth in all its forms is highly essential. However, the closer mathematical efforts can be brought to a material universe, the greater will be our understanding and knowledge of nature. Likewise the physicist, the chemist, the biologist, the engineer, and a host of others must to a large degree furnish the raw material which is to be used and treated by the logical processes of mathematics. New patterns must be laid out and in so far as possible according to designs capable of mathematical treatment.

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